Short and Long Interest Rate Targets*

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Abstract

Can both short and long-term interest rates be targeted independently? Can the target of the term structure help solve the problem of multiplicity of equilibria

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that occurs when only the short rate is targeted? Both questions are addressed, and the answer is yes to both.

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1 **Introduction**

The low targets for short-term nominal interest rates during the recent financial crisis, very close to zero, prompted again the policy question of whether a central bank can target both short and long-term rates, with the hope of lowering the latter, given that the former cannot be lowered. The recent crisis has provided empirical support for this ability of a central bank, to target rates at different maturities. The Federal Reserve may have been able to influence the long-term rates, through the Quantitative Easing 1, 2 and 3 programs in 2008-2009, late 2010 and 2012.\(^1\) In 2009, the European Central Bank (ECB) conducted one week, three and six months, and one year, liquidity providing operations at fixed rates. In 2011, the ECB started to lend at fixed rates at even longer horizons.\(^2\) And there is further historical evidence that "...a sufficiently determined Fed can peg or cap Treasury bond prices and yields at other than the shortest maturities."\(^3\) In the 40’s and 50’s, before the Federal Reserve-Treasury Accord of 1951, the Fed established both the rate on the 90-day Treasury bill and a ceiling on the 12-month Treasury certificate. Operation Twist, in the 1960’s, was also an attempt by the Fed to raise short rates and lower long rates.

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\(^2\)The one week operations are the main refinancing operations (MROs). The operations at long horizons are the long term and very long term refinancing operations, LTROs and VLTROs (three years).

\(^3\)From a speech by Ben Bernanke to the National Economists Club in 2002.
While there is empirical evidence for the ability of a central bank to target interest rates at different maturities, there is no theoretical basis for it.\textsuperscript{4} This is the contribution of this paper. Can both short and long-term interest rates be targeted independently? Can the target of the term structure help solve the problem of multiplicity of equilibria that occurs when only the short rate is targeted? We address both questions, and the answer is yes to both.

The problem of multiplicity of equilibria when the monetary policy instrument is the short-term, nominal interest rate was first formally addressed by Sargent and Wallace (1975). They consider an ad hoc macro model with rational expectations, assume that the policy rate responds to historical values of exogenous and endogenous variables, and show that the price level is indeterminate. Nakajima and Polemarchakis (2003) take an approach closer to ours. They consider a cash-in-advance model with uncertainty and assume that policy is a target for the interest rate on one period nominal bonds. They compute the degrees of multiplicity. In the deterministic model there is one degree of multiplicity, say, the initial price level. Instead, when uncertainty is taken into account, there is one degree for each possible history.

This multiplicity of equilibria under uncertainty, when policy targets the short rates, is the reason for our results. It is, indeed, because there are multiple equilibrium values for the price level under a target for the short rate, that there are also multiple equilibrium values for the long-term nominal interest rates. For this reason, short and long rates are independent monetary policy instruments.

A target of both short and long rates is equivalent, under general conditions, to a target of the returns on state-contingent nominal assets. If policy were to target those rates of return, it would be able to pin down the price level in every state, for a given

\textsuperscript{4}Recent, independent work by Magill and Quinzii (2014) has results similar to ours. The focus is different: We focus on the ability of the central bank to target both short and long rates. They focus on the possibility of using the term structure to anchor expectations.
Sargent and Wallace (1975) and Nakajima and Polemarchakis (2003) do not consider interest rate feedback rules in which the policy rate can respond to contemporaneous endogenous variables or expectations of future variables. But considering those feedback rules, in general, does not solve the multiplicity problem (see Loisel (2009) and Adão, Correia and Teles (2011) for exceptions). While it is possible to design Taylor-type rules to ensure local determinacy, globally there are still many equilibria. The conditions for local determinacy may in fact be conditions for global indeterminacy, as shown by Benhabib, Schmitt-Grohe and Uribe (2001), among others.

Eggertsson and Woodford (2003) also address the question of whether it is possible, or useful, for policy to affect long-term rates. In their model, for each equilibrium in prices and quantities, there are multiple portfolio compositions that support the equilibrium. This implies that changes in the relative supply of bonds of different maturities do not necessarily affect the set of equilibria. But it does not mean that the target of the prices on those assets will not affect the particular equilibrium that is implemented, as we show it does.

We start by illustrating, using a simple flexible price monetary model (described in section 2), that targeting the return on noncontingent short-term bonds cannot pin down the distribution of realized inflation across states (section 3.2). The nominal interest rate is a noncontingent return and, therefore, it only imposes restrictions on a conditional expectation of inflation. Realized inflation affects the marginal utility of money, and term premia are a function of the covariance between that marginal utility and the price of the noncontingent nominal assets. Since realized inflation is not pinned down, those covariances aren’t either, and nominal term premia are not pinned down. Instead, under sticky prices, the multiplicity is for both allocations and prices, and a target for the state contingent interest rates pins down a unique equilibrium given an initial allocation.

\footnote{This result is in a model with flexible prices. Instead, under sticky prices, the multiplicity is for both allocations and prices, and a target for the state contingent interest rates pins down a unique equilibrium given an initial allocation.}
uniquely determined.

We show that the target of the short and long-term interest rates can solve the multiplicity of equilibria associated with uncertainty (section 3.4). The intuition is simple: The targeting of the term structure imposes restrictions on the term premia and therefore on the distribution of prices across states. If there were as many contingencies as restrictions, the distribution of price levels would be uniquely pinned down, for a given initial price level. An alternative intuition uses an equivalence between targeting the prices of one-period-ahead, state-contingent nominal assets and the prices of the noncontingent nominal assets of different maturities. If monetary policy were to target the prices of the state-contingent nominal assets, then given an initial price level, it would be able to pin down the price level in every date and state (section 3.3).

In order to target the nominal term structure, the government or central bank stands ready to buy and sell any quantity of bonds of different maturities at fixed rates. While both short and long rates can be used to determine uniquely the equilibrium for the price level and the real allocations, the supply of bonds of different maturities is not uniquely pinned down. In particular, the actual supply of noncontingent bonds of different maturities can be zero in equilibrium (section 4.1). Combinations of contingent assets and taxes can play the same role of the supply of bonds of different maturities.

In sections 4.3, the results are interpreted by discussing the role of nonfundamental uncertainty for policy, and in section 4.4, the results are compared to the ones in the literature on local and global determinacy. There is also a discussion of the role of the cash-in-advance constraint in section 4.2. The results are extended to an environment with sticky prices in section 5. In section 6 we provide a simple example with a cashless economy.
2 A model with flexible prices

The economy consists of a representative household, a representative firm behaving competitively, and a government. The uncertainty in period \( t \geq 0 \) is described by the random variable \( s_t \in S_t \), where \( S_t \) is the set of possible events at \( t \), and the history of its realizations up to period \( t \) (state or node at \( t \)), \( (s_0, s_1, ..., s_t) \), is denoted by \( s^t \in S^t \). The distribution of \( s_t \) is discrete. The number of states in period \( t \geq 0 \) is \( \Phi_t \). There is fundamental uncertainty if technology and government spending \( A(s^t) \) and \( G(s^t) \) vary with the state \( s^t \). Otherwise uncertainty is nonfundamental.

Production uses labor according to a linear technology. A cash-in-advance constraint is imposed on the households’ transactions with the timing structure described in Lucas and Stokey (1983). That is, each period is divided into two subperiods, with the assets market operational in the first subperiod and the goods market in the second.

2.1 Competitive equilibria

**Households** The households have preferences over consumption \( C(s^t) \), and leisure \( L(s^t) \), described by the expected utility function

\[
U = E_0 \left\{ \sum_{t=0}^{\infty} \beta^t u \left( C(s^t), L(s^t) \right) \right\},
\]

where \( E_t \) is the expectation conditional on the information in state \( s^t \) and \( \beta \) is a discount factor. The households start period \( t \), in state \( s^t \), with nominal wealth \( \bar{W}(s^t) \). They decide to hold money, \( M(s^t) \), and to buy \( B^j(s^t) \) nominal, noncontingent, government bonds that pay \( R^j(s^t)B^j(s^t), j = 1, ..., m \), periods later. \( R^t(s^t) \) is the gross short-term nominal interest rate. They also buy \( Z(s^{t+1}) \) units of one-period-ahead, state-contingent, nominal government securities. Each security pays one unit of money at the
beginning of period $t+1$ in state $s^{t+1}$. Let $Q(s^{t+1}/s^t)$ be the beginning of period $t$ price of these securities normalized by the probability of the occurrence of the state. The households spend $E_t Q(s^{t+1}/s^t) Z(s^{t+1})$ in state-contingent nominal securities. Thus, in the assets market at the beginning of period $t$ they face the constraint

$$M(s^t) + \sum_{j=1}^{m} B^j(s^t) + E_t Q(s^{t+1}/s^t) Z(s^{t+1}) \leq \mathbb{W}(s^t), \quad (2)$$

where the initial nominal wealth $\mathbb{W}(s^0)$ is given.

Consumption must be purchased with money, in the goods market, according to the cash-in-advance constraint

$$P(s^t) C(s^t) \leq M(s^t). \quad (3)$$

At the end of the period, the households receive the labor income $W(s^t) N(s^t)$, where $N(s^t) = 1 - L(s^t)$ is labor and $W(s^t)$ is the nominal wage rate and pay lump sum taxes $T(s^t)$. Thus, the nominal wealth that households bring to state $s^{t+1}$ is

$$\mathbb{W}(s^{t+1}) = M(s^t) + \sum_{j=1}^{m} R^j(s^{t+1-j}) B^j(s^{t+1-j}) + Z(s^{t+1}) - P(s^t) C(s^t) + W(s^t) N(s^t) - T(s^t) \quad (4)$$

The households’ problem is to maximize expected utility (1) subject to the restrictions (2), (3), (4), together with a no-Ponzi games condition on the holdings of assets.

The following are first order conditions of the households’ problem:

$$\frac{u_L(s^t)}{u_C(s^t)} = \frac{W(s^t)}{P(s^t) R^1(s^t)}, \quad (5)$$
\[
\frac{u_C(s^t)}{P(s^t)} = R^j(s^t) E_t \left[ \frac{\beta u_C(s^{t+j})}{P(s^{t+j})} \right], \quad j = 1, \ldots, m, 
\]

(6)

\[
Q(s^{t+1}/s^t) = \beta \frac{u_C(s^{t+1})}{u_C(s^t)} \frac{P(s^t)}{P(s^{t+1})}. 
\]

(7)

Condition (5) sets the intratemporal marginal rate of substitution between leisure and consumption equal to the real wage adjusted for the cost of using money, \(R^1(s^t)\). Condition (6) is an intertemporal marginal condition for the optimal choice of risk-free nominal bonds of different maturities. Condition (7) equates the price of one unit of money\(^6\) at time \(t + 1\), for each state \(s^{t+1}\), in units of money at time \(t\), in state \(s^t\), to the intertemporal marginal rate of substitution.

**Firms** The firms are competitive and prices are flexible. The production function of the representative firm is linear \(Y(s^t) = A(s^t) N(s^t)\). It follows that the equilibrium real wage is

\[
\frac{W(s^t)}{P(s^t)} = A(s^t). 
\]

(8)

**Government** The policy variables are taxes, \(T(s^t)\), nominal interest rates of different maturities, \(R^j(s^t)\), state-contingent nominal prices, \(Q(s^{t+1}/s^t)\), money supplies, \(M(s^t)\), noncontingent public debt, \(B^j(s^t)\) and state-contingent debt, \(Z(s^{t+1})\). The government expenditures, \(G(s^t)\), are exogenous.

Imposing the terminal condition \(\lim_{T \to \infty} E_t Q(s^{T+1}) \mathbb{W}(s^{T+1}) = 0\), the budget constraints of the government can be written as

\(^6\)The prices \(Q(s^{t+1}/s^t)\) are the actual prices divided by the conditional probability of occurrence of state \(s^{t+1}\).
\[
\sum_{s=0}^{\infty} E_t Q(s^{t+1}/s^t) [M(s^{t+1}) (R^1(s^{t+1}) - 1) + T(s^{t+1}) - P(s^{t+1}) G(s^{t+1})] = W(s^t),
\]

(9)

where \( Q(s^{t+1}) = Q(s^{t+1}/s_0) \) is the price of one unit of money in the beginning of period \( t + 1 \), in state \( s^{t+1} \), as of period 0, with \( Q(s_0) = 1 \).

Market clearing  
The goods and labor market clearing conditions are \( C(s^t) + G(s^t) = A(s^t) N(s^t) \) and \( N(s^t) = 1 - L(s^t) \). Market clearing in the money and bond markets has been implicitly imposed.

Equilibria  
A competitive equilibrium for the sequences of quantities, prices and policies, \( \{C(s^t), L(s^t)\}, \{P(s^t), R^j(s^t), Q(s^{t+1}/s^t), M(s^t), B^j(s^t), Z(s^{t+1})/s^t, T(s^t)\} \), must satisfy the following conditions: The resource constraints

\[
C(s^t) + G(s^t) = A(s^t) (1 - L(s^t)),
\]

(10)

the intratemporal conditions

\[
\frac{u_C(s^t)}{u_L(s^t)} = \frac{R^1(s^t)}{A(s^t)},
\]

(11)

obtained from the households intratemporal conditions (5) and the firms optimal conditions (8), as well as the cash-in-advance constraints (3) holding with equality as long as \( R^1(s^t) > 1 \), the intertemporal conditions (6) and (7), and the budget constraints (9). Because nominal interest rates cannot be negative in equilibrium, it must also be the case that \( E_t[Q(s^{t+1}/s^t)\cdots Q(s^{t+j}/s^{t+j-1})] \geq 1 \) for \( j \geq 1 \). Notice that, given the nominal interest rate \( R^1(s^t) \), the consumption and leisure allocation in each state \( s^t \), \( C(s^t) \) and \( L(s^t) \), is uniquely determined by the resource constraint (10) and the intratemporal condition (11), for each state \( s^t \).
The equations identified above determine a set of equilibrium allocations, prices and policy variables. We now take a subset of those equations to restrict a subset of the equilibrium variables. They are the necessary and sufficient conditions on the equilibrium price levels, rates of return on the bonds of different maturities and the prices of the state-contingent bonds.

2.2 Summarizing the competitive equilibrium conditions

From the resource constraints (10), and the intratemporal conditions (11), consumption and leisure can be written as functions of the short-term nominal rate, \( C(R^1(s^t)) \) and \( L(R^1(s^t)) \). This means that in this model with flexible prices, if the short-term nominal interest rate \( R^1(s^t) \) is set exogenously in every date and state, then the allocation is pinned down uniquely.\(^7\)

Let \( u_C(R^1(s^t)) \equiv u_C(C(R^1(s^t)), L(R^1(s^t))) \). The marginal conditions for the non-contingent bonds can then be written as

\[
\frac{u_C(R^1(s^t))}{P(s^t)} = R^j(s^t) \left[ \beta^j u_C(R^1(s^{t+j})) \right], \quad t \geq 0, \ j = 1, \ldots, m, \quad (12)
\]

and the marginal conditions for the prices of the state-contingent bonds, as

\[
Q(s^{t+1}/s^t) = \beta \frac{u_C(R^1(s^{t+1}))}{u_C(R^1(s^t))} \frac{P(s^t)}{P(s^{t+1})}, \quad (13)
\]

They imply

\[
E_t \left[ Q(s^{t+1}/s^t) \ldots Q(s^{t+j}/s^{t+j-1}) \right] = \frac{1}{R^j(s^t)}, \quad j = 1, \ldots, m. \quad (14)
\]

\(^7\)If the economy was cashless, the allocation would not depend on the interest rate and therefore, it would always be unique. This case will be discussed in section 4.2.
Conditions (12) and (13), together with a condition of nonnegativity of nominal interest rates, that can be written as
\[ \frac{u_C(R^1(s^t))}{P(s^t)} \geq 1 \text{ for } j \geq 1, \]
summarize the competitive equilibrium restrictions on the rates of return of the noncontingent bonds for the different maturities, \( \{R^j(s^t)\}, j = 1, \ldots, m, \) the prices of the state-contingent bonds, \( \{Q(s^{t+1}/s^t)\} \), and the price levels, \( \{P(s^t)\} \). They are the set of implementability conditions for those variables. The other equilibrium conditions restrict the other variables \( \{M(s^t), B^j(s^t), Z(s^{t+1})\} \) and \( \{T(s^t)\} \). The cash-in-advance constraints, (3), restrict \( M(s^t) \), the budget constraints, (9), can be satisfied by appropriate choices of lump sum taxes, \( T(s^t) \), and levels of assets, \( B^j(s^t) \) and \( Z(s^{t+1}) \). There are multiple ways of satisfying the budget constraints with those variables.

The marginal conditions for the noncontingent bonds, (12), for maturities \( j = 2, \ldots, m \), can be written as
\[
\frac{u_C(R^j(s^t))}{P(s^t)} = R^1(s^t) E_t \left[ \frac{\beta u_C(R^1(s^{t+1}))}{R^{j-1}(s^{t+1}) P(s^{t+1})} \right], \quad j = 2, \ldots, m. \tag{15}
\]

These can be used, together with (12) for \( j = 1 \), to obtain the following no arbitrage conditions between bonds of different maturities:
\[
\frac{R^j(s^t)}{R^1(s^t)} = E_t \left[ R^{j-1}(s^{t+1}) \right] + \frac{\text{cov}_t \left[ R^{j-1}(s^{t+1}), E_{t+1} \left[ \frac{u_C(R^1(s^{t+j}))}{P(s^{t+j})} \right] \right]}{E_t \left[ \frac{u_C(R^1(s^{t+j}))}{P(s^{t+j})} \right]}, \quad j = 2, \ldots, m, \tag{16}
\]
which are the standard conditions for the term premia in models that are not linearized.\(^8\)

In the following section we specify policy and analyze the multiplicity of equilibria.

\(^8\)The condition is obtained directly from \( R^j(s^t) E_t \left[ \frac{\beta u_C(R^1(s^{t+j}))}{P(s^{t+j})} \right] = \beta R^1(s^t) E_t \left[ R^{j-1}(s^{t+1}) E_{t+1} \left[ \frac{\beta^{-1} u_C(R^1(s^{t+j}))}{P(s^{t+j})} \right] \right], \quad j = 2, \ldots, m. \)
With a policy that targets the short rate, there are multiple equilibria for the price level. This implies the main result of this paper that can be anticipated from equation (16). Since there are multiple stochastic processes for the price level $P(s^{t+1})$, there are also multiple term premia, and therefore multiple rates of return on bonds of different maturities. Targeting the term structure imposes the restrictions on those term premia that are necessary to determine the stochastic process for the price level.

3 Policy and multiplicity of equilibria

Monetary policy is first assumed to be a target for the short term nominal interest rate, $R^1(s^t)$. In this case, the path for the price level, $P(s^t)$, is restricted only by (12) for $j = 1$. The other equations restrict the other variables: Given a process for the price level, $P(s^t)$, conditions (12) for $j = 2, ..., m$, and (13) determine, respectively, $R^j(s^t)$ for $j = 2, ..., m$ and $Q(s^{t+1}/s^t)$.

3.1 No uncertainty

Without uncertainty, condition (12) becomes

$$\frac{u_C(R^1_t)}{P_t} = \beta R^1_r \frac{u_C(R^1_{t+1})}{P_{t+1}}, t \geq 0. \quad (17)$$

In a world without uncertainty, given an initial price level, $P_0$, this condition would determine recursively the price level uniquely for every date $t \geq 1$. This is a standard textbook result. It is the indeterminacy of the initial price level that most of the literature focuses on. Most of the analysis in the indeterminacy literature is in deter-

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9 For simplicity of exposition, we ignore the implementability condition $\frac{u_C(R^1_t)}{P_t} \geq 1$ for $j = 1$, that ensures the nonnegativity of nominal interest rates.
ministic models, and there, for a given initial price level, there is a single path for the price level in subsequent periods. We have nothing to say about this indeterminacy. For the remaining of the paper, the initial price level is taken as given.

3.2 Multiplicity of equilibria with a target for the short rate

Under uncertainty, even if the initial price level is given, there are still multiple equilibria for the price level in each state. To see this, notice from (12), that in any period $t \geq 1$, given $P(s^{t-1})$, there are $\Phi_{t-1}$ equations to determine $\Phi_t$ variables, $P(s^t)$, where, again, $\Phi_t$ is the number of states at $t$. More specifically, for each state $s^{t-1}$, there is one equation to determine $\#S_t$ variables, where $\#S_t$ is the number of possible events at $t$. The indeterminacy of the initial price level in the deterministic economy becomes the indeterminacy of one price level per history, under uncertainty. This result of multiplicity of equilibria with a target for the one period nominal interest rate is in Nakajima and Polemarchakis (2003).

This explosion in the degrees of multiplicity under uncertainty results from pegging the noncontingent nominal interest rate, instead of the returns on state-contingent nominal assets. If, instead, these were pegged, there would be a single degree of multiplicity as in the deterministic case. Our results on the term structure are obtained because a peg of the prices of the one-period-ahead, state-contingent nominal assets is, under general conditions, equivalent to a peg of the returns on noncontingent bonds of different maturities.

\footnote{There is also multiplicity of equilibria when interest rate policy follows a Taylor-type rule. In that case there is, in general, one determinate equilibrium, but many equilibria globally. A money supply rule in this model would also not implement a unique equilibrium.}
3.3 Targeting the returns on state-contingent nominal assets

In these monetary economies in which policy is a target for the interest rate on a short term, noncontingent nominal asset, expected inflation is pinned down, but realized inflation is not. For each policy on the short rate, there is a single allocation, but multiple equilibria for the price level, even given an initial price level.

Each of those equilibria can be implemented uniquely with a target of the returns on the one-period-ahead, state-contingent nominal assets. To show this result, that a target of \( Q(s^{t+1}/s^{t}) \) determines the price level, \( P(s^{t}) \), for \( t \geq 1 \), for each \( s^{t} \), given the initial price, \( P(s^{0}) \), notice that, in this case, the implementability conditions on \( Q(s^{t}/s^{t-1}) \), \( R_{1}(s^{t}) \) and \( P(s^{t}) \) are (13), together with (14) for \( j = 1 \). The other conditions, (12) for \( j = 2, ..., m \), determine the returns on the noncontingent bonds of longer maturities.

**Proposition 1** If the returns on one-period-ahead, state-contingent, nominal assets are set exogenously for every date and state, given an initial price level, there is a unique equilibrium for the allocations and prices.

Proof: Let \( P(s^{0}) \) be given. Given the target for \( Q(s^{t+1}/s^{t}) \), \( t \geq 0 \), \( R_{1}(s^{t}) \), \( t \geq 0 \) are determined uniquely from (14), and given \( P(s^{t-1}) \), \( P(s^{t}) \) is obtained from the intertemporal conditions (13) for \( t \geq 1 \).

Notice that this policy implements uniquely each equilibrium in the set of possible equilibria associated with a target of the short rate. Indeed, for each equilibrium for the price level restricted by (12) for \( j = 1 \), there are prices \( Q(s^{t}/s^{t-1}) \) satisfying (13) and (14) for \( j = 1 \) that will implement it.
3.4 Targeting the term structure

An alternative way to target the one-period-ahead, state-contingent interest rates is a target for the returns on noncontingent nominal assets of different maturities. It follows that it is possible to deal with the multiplicity of equilibria associated with uncertainty with a target of the term structure. It is also an implication that short and long rates are independent policy instruments.

We first show an equivalence between the two targets. The prices of the noncontingent assets of different maturities, \( \frac{1}{R^j(s^t)} \), are determined by arbitrage from the prices of the state-contingent assets as in (14). Given a target for the state-contingent prices, there is a unique term structure. The more intriguing question is whether the reverse is also true.

In order to show this, it is useful to write the no arbitrage conditions for assets of maturity \( j = 2, ..., m \), with \( m = n \), as

\[
\frac{1}{R^j(s^t)} = E_t \left[ \frac{Q(s^{t+1}/s^t)}{R^{j-1}(s^{t+1})} \right], j = 2, ..., n,
\]

which is obtained from (15), using (13).

Let \( \pi(s^{t+1}/s^t) \) be the probability of state \( s^{t+1} \) conditional on state \( s^t \). Using (18), the prices of the state-contingent assets are obtained uniquely from the prices of the noncontingent assets, according to

\[
\begin{bmatrix}
Q((s^t, s_1)/s^t) \\
Q((s^t, s_2)/s^t) \\
... \\
Q((s^t, s_n)/s^t)
\end{bmatrix}
= M^{-1}
\begin{bmatrix}
\frac{1}{R^1(s^t)} \\
\frac{1}{R^2(s^t)} \\
... \\
\frac{1}{R^n(s^t)}
\end{bmatrix},
\]

(19)
with
\[
M = \begin{bmatrix}
\pi \left( \frac{\left( s_t^t, s_1 \right)}{s_t^t} / s_t^t \right) & \pi \left( \frac{\left( s_t^t, s_2 \right)}{s_t^t} / s_t^t \right) & \cdots & \pi \left( \frac{\left( s_t^t, s_n \right)}{s_t^t} / s_t^t \right) \\
\frac{\pi \left( s_t^t, s_1 / s_t^t \right)}{R^1(s_t^t,s_1)} & \frac{\pi \left( s_t^t, s_2 / s_t^t \right)}{R^1(s_t^t,s_2)} & \cdots & \frac{\pi \left( s_t^t, s_n / s_t^t \right)}{R^1(s_t^t,s_n)} \\
\cdots & \cdots & \cdots & \cdots \\
\frac{\pi \left( s_t^t, s_1 / s_t^t \right)}{R^{n-1}(s_t^t,s_1)} & \frac{\pi \left( s_t^t, s_2 / s_t^t \right)}{R^{n-1}(s_t^t,s_2)} & \cdots & \frac{\pi \left( s_t^t, s_n / s_t^t \right)}{R^{n-1}(s_t^t,s_n)}
\end{bmatrix},
\] (20)

provided the matrix $M$ is invertible. The proposition follows.\textsuperscript{11}

**Proposition 2** Let $S_{t+1} = \{s_1, s_2, ..., s_n\}$, $t \geq 0$, and suppose there are nominal non-contingent assets of maturity $j = 1, ..., m$. Let $m \geq n$. The prices of the one-period-ahead, state-contingent assets are obtained uniquely with a target for the returns on $n$ of the nominal noncontingent assets, provided the matrix $M$ in (20) is invertible. The invertibility of $M$ holds generally.

It is a necessary condition for the matrix $M$ to be invertible that there be variability in the nominal rates. In particular, the interest rates for each maturity $j = 1, ... n - 1$, cannot be constant across states. They must vary across states in ways that are not perfectly correlated. When $n = 2$, the matrix is invertible if $R^1(s^t, s_1) \neq R^1(s^t, s_2)$. The difference between the two rates, can be arbitrarily small, though. Similarly, an arbitrarily small change in the rates can insure a nonzero determinant, and therefore the invertibility of the matrix. This means that the invertibility of the matrix $M$ holds generally.

The number of maturities $m$ cannot be lower than the number of contingencies $n$. It must be that $n$ maturities are pegged independently in a way that guarantees that the $nxn$ matrix of coefficients described above is invertible. As the number of contingencies $n$ is increased and becomes arbitrarily large, the number of maturities would also be

\textsuperscript{11}This result is similar to the one obtained in Angeletos (2002) and Buera and Nicolini (2004) on replicating state-contingent public debt with debt of different maturities. Their emphasis is on quantities, ours on prices, but the mechanisms are similar.
arbitrarily large. In the limit, the whole term structure would have to be targeted. Targeting the term structure is therefore, under general conditions, equivalent to a target of the state-contingent prices.

From proposition 2, it follows that targeting the term structure implies the targeting of the state-contingent returns. From proposition 1, a target for the state-contingent returns implements a unique equilibrium. It then follows that:

**Corollary 3** Let $S_{t+1} = \{s_1, s_2, \ldots, s_n\}$ and suppose there are nominal noncontingent assets of maturity $j = 1, \ldots, m$. Let $m \geq n$. If the returns on $n$ of these assets are set exogenously, then, in general, there is a unique equilibrium for the allocations and prices, given the initial price level $P(s_0)$.

In the particular cases in which the matrix $M$ is not invertible, the interest rate for at least one maturity can be written as a linear combination of rates at other maturities. In those cases, the interest rates for the $n$ different maturities would not be independent targets. But, because the conditions for invertibility are general, the interest rates at different maturities are indeed independent instruments. This is stated in the following corollary.

**Corollary 4** Short and long-term nominal interest rates are independent monetary policy instruments.

While the question of using long rates as an instrument of policy is typically raised when short rates are very close to the zero lower bound, it turns out that, at the zero bound, the conditions for the results in proposition 2 and corollary 3 may not be verified.

If the economy was always at the zero bound, in every date and state, $R^1(s^t) = 1$ for all $t$ and $s^t$, the matrix $M$ would not be invertible. Another way to see this, is that
the covariance in the no arbitrage condition (16), would be zero for any process of the
price level. This means that targeting the longer rates would not impose restrictions
on the price level, and it also means that the longer rates would be uniquely obtained
from the short rates. Indeed, from (16), the expectations hypothesis would trivially
hold, and \( R^j (s^t) = 1 \) for all \( j \) and \( s^t \).

At the zero bound, if the economy was to remain there forever, the condition of
effect variability in the interest rates would not be satisfied. Short and long rates
would not be independent policy instruments and the process for the price level would
not be pinned down. It turns out, however, that, as mentioned above, an \( \varepsilon \) deviation
from the zero bound would allow to recover the results.

If the economy was temporarily at the zero bound, then short and long maturities,
but not medium-term, would still be independent targets. The returns on medium
maturities would be obtained from the shorter rates, but longer rates could still be
targeted independently. To see this, suppose, for instance, that the economy was at
the zero bound in periods \( t \) and \( t + 1 \) for all possible contingencies, meaning that
\( R^1 (s^t) = R^1 (s^{t+1}) = 1 \). Then, the two period return at \( t \) would be given by \( R^2 (s^t) =
R^1 (s^t) R^1 (s^{t+1}) = 1 \), so that it would not be a separate instrument. But suppose there
was a target for the returns on the one and three-period bonds, \( R^1 (s^t) \) and \( R^3 (s^t) \). And
suppose, again for simplicity, that there were only two contingencies, \( s_1 \) and \( s_2 \). Then,
using the one period and three period maturity bonds, it follows that
\[
\begin{bmatrix}
\frac{1}{R^3(s^t)} \\
\frac{1}{R^4(s^t)}
\end{bmatrix}
= 
\begin{bmatrix}
\pi \left( \frac{(s^t, s_1) / s^t}{R^2(s^t, s_1)} \right) & \pi \left( \frac{(s^t, s_2) / s^t}{R^2(s^t, s_2)} \right) \\
\pi \left( \frac{(s^{t+1}, s_1) / s^{t+1}}{R^2(s^{t+1}, s_1)} \right) & \pi \left( \frac{(s^{t+1}, s_2) / s^{t+1}}{R^2(s^{t+1}, s_2)} \right)
\end{bmatrix}
\begin{bmatrix}
Q \left( \frac{(s^t, s_1) / s^t}{R^3(s^t, s_1)} \right) \\
Q \left( \frac{(s^t, s_2) / s^t}{R^3(s^t, s_2)} \right)
\end{bmatrix}.
\]
If \( R^2 (s^t, s_1) \neq R^2 (s^t, s_2) \neq 1 \),
then the longer rate \( R^3 (s^t) \) could be targeted independently of the shorter rate \( R^1 (s^t) \).
There would be a single solution for the state-contingent prices, and therefore, it would
be possible to determine the price levels \( P(s^t, s_1) \) and \( P(s^t, s_2) \).
4 Further on implementation

In this section, further issues of implementation are discussed. The first issue is the role of the relative supply of bonds of different maturities for the implementation of equilibria.

4.1 The equilibrium level of debt can be zero

In targeting the interest rates on the assets of different maturities, the government, or central bank, stands ready to supply and demand any quantity of those bonds that may be desired by the private sector, at given rates. This implements a unique equilibrium for the price level, consumption and labor. But it does not require a particular equilibrium quantity of those noncontingent assets. In our model, the level of those assets is not uniquely determined. In particular, it can be zero in equilibrium.

The only equilibrium conditions restricting the quantity of the noncontingent bonds are the government budget constraints

\[
\sum_{s=0}^{\infty} E_t Q \left( s^{t+s+1}/s^t \right) \left[ M \left( s^{t+s} \right) \left( R^1 \left( s^{t+s} \right) - 1 \right) + T \left( s^{t+s} \right) - P \left( s^{t+s} \right) G \left( s^{t+s} \right) \right] = \mathbb{W} \left( s^t \right), \text{ for all } t = 1, 2..., \text{ and all } s^t, \tag{21}
\]

where \( \mathbb{W} \left( s^t \right) = M \left( s^{t-1} \right) + Z \left( s^t \right) + \sum_{j=1}^{m} R^j \left( s^{t-j} \right) B^j \left( s^{t-j} \right) + P \left( s^{t-1} \right) G \left( s^{t-1} \right) - T \left( s^{t-1} \right) \).

These conditions also restrict the path for the taxes, as well as the quantity of the state-contingent debt. In particular, for each path of \( Q \left( s^{t+1}/s^t \right), R^j \left( s^t \right), M \left( s^t \right), P \left( s^t \right), G \left( s^t \right), \) for \( t \geq 0, \) there are multiple ways of satisfying the government budget constraint for each date and state, with taxes, state-contingent debt, or with debt of debt of

\[12\] The reason why the budget constraints considered here are for \( t \geq 1 \) is that \( \mathbb{W} \left( s^t \right) \) is exogenous.
different maturities. It is straightforward to see that the level of the noncontingent debt of different maturities is not pinned down. Given a path of \( \{B^j(s^t)\}_{t=0}^{\infty} \), each condition (21) in period \( t \geq 1 \), and state \( s^t \), can be satisfied with some \( T(s^t) \) or \( Z(s^t) \), \( t \geq 1 \). And there is no other equilibrium condition where those variables play a role, except for the budget constraint at \( t = 0 \), but that constraint can be satisfied with the tax in period 0, \( T(s^0) \).

One particular implementation of the equilibrium is with zero quantity of the non-contingent bonds. Either the taxes or the state-contingent debt could play the role of the debt of different maturities.

In this economy, while the direct target of the returns on the noncontingent assets of different maturities can implement a unique equilibrium for the allocation and price levels, there are still multiple portfolio compositions supporting that equilibrium. In order for the quantity of noncontingent bonds of different maturities to be uniquely determined, additional restrictions must be imposed on taxes and state-contingent debt. These results are related to the ones obtained by Eggertsson and Woodford (2003) or, more generally, to the irrelevance results in Wallace (1981).

### 4.2 The cashless economy

In the analysis above, money played no direct role in the implementation of equilibria. Policy was a target for the interest rates on bonds of different maturities in the sense that the government or central bank stood ready to buy or sell any quantity of those bonds at those rates. This policy was able to implement a unique equilibrium for the quantities of consumption and labor and the price level, given the initial level of prices. As seen in the previous section, there are multiple supplies of the noncontingent bonds supporting an equilibrium, since state-contingent debt or variable (lump sum) taxes can play the same role as debt of different maturities. Instead, there is only one
equilibrium level for nominal money balances, the one that satisfies the cash-in-advance constraint, (3), provided the short rate is not exactly zero. But, again, this plays no role in the implementation, as neither did the supply of bonds of different maturities.

That there is no direct role for money can be seen clearly by assuming, as is standard in the new keynesian literature, that the economy is cashless (see Woodford, 2003). One way to frame this is to assume that, instead of the cash-in-advance restriction, the utility function was additive in the utility from real money balances.\(^{13}\) In this case, the intratemporal conditions would be \(\frac{u_C(s^t)}{u_L(s^t)} = \frac{1}{A(s^t)}\),\(^{14}\) which, together with the resource constraints (10), would determine consumption, \(C(s^t)\), and leisure, \(L(s^t)\), in each date and state. Consumption and leisure would depend on \(A(s^t)\) and \(G(s^t)\), but not on the nominal interest rate, \(R^1(s^t)\). The remaining of the analysis would go through exactly as above. There would be multiple equilibrium distributions of price levels for a target of the short rate. There would be a single equilibrium for the price level if the target was for the state-contingent interest rates, and this could be achieved with a target for the term structure.

### 4.3 Multiple equilibria and sunspots

Monetary policy can be used to implement a unique equilibrium, again, given the initial price level. Now, what are the properties of these unique equilibria? In particular, can an equilibrium be isolated from nonfundamental uncertainty? More generally, we want to analyze, in the context of this model, what is the role of nonfundamental uncertainty for monetary policy.

In the way uncertainty has been modelled, with an underlying (nonfundamental)\(^{13}\)Literally, the economy is cashless when the weight of that term in the utility function tends to zero.\(^{14}\)In the model with a cash-in-advance, the analogous conditions to these are (11) with the short term nominal interest rate acting as a distortion.
random variable $s^t$, there is fundamental uncertainty when technology and government
consumption vary with $s^t$. Here, uncertainty is going to be modelled differently. We
first assume that there is only fundamental uncertainty, described by $\tilde{s}_t = (A_t, G_t)$,
where $A_t$ and $G_t$ are the stochastic technology parameter and government consumption,
respectively. The history of fundamental shocks is $\tilde{s}_t$.

This is a particular case of the analysis above, so clearly all the results go through.
If policy is a target for the noncontingent, short term, nominal interest rate, $R^1(\tilde{s}_t)$,
then, as shown before, there are multiple equilibria. The distribution of price levels
across states of the world is not pinned down. It is pinned down, if policy is a target for
$Q(\tilde{s}^{t+1}/\tilde{s}^t)$. Similarly, if policy is a target for the term structure, under the conditions
specified above, there is also a unique equilibrium. By construction, the equilibria do
not depend on nonfundamental uncertainty.

We can now close down the fundamental uncertainty and only consider a sunspot,
$\tilde{s}_t = \Xi_t$. The analysis is the same, but the interpretation is different. Here, the target
for the state-contingent prices, $Q(\tilde{s}^{t+1}/\tilde{s}^t)$, must be done for all the nonfundamental
contingencies.\footnote{This may be a daring task in practice, since a sunspot is, by definition, an arbitrary random variable.} And in principle, the unique equilibrium that is implemented could
depend on the sunspot. That would be the case if the prices $Q(\tilde{s}^{t+1}/\tilde{s}^t)$ differed across
states. But, if those prices were the same for all states, then the equilibrium would
not vary with the sunspot. It would still be necessary to target the price for every
contingency, but the equilibrium would not depend on the contingency.\footnote{A simple analog to this in a real model is when distortionary taxes can vary with a sunspot variable. If taxes are set exogenously, then there is a unique equilibrium. That equilibrium does not depend on the sunspot as long as taxes do not vary with it.}

More generally, in order for policy to implement a unique equilibrium, the prices
of the state-contingent nominal assets must be set for any possible realization of both
fundamental and nonfundamental uncertainty. When those prices are the same for all
realizations of the nonfundamental uncertainty, the equilibrium will be isolated from sunspots.

Does this analysis go through when instead of a target for the state-contingent prices, policy is a target for the term structure? It does go through, but there is a difference that may be worth pointing out. Again, for simplicity, suppose that the only source of uncertainty is nonfundamental. If policy is a target for the rates on the bonds of different maturities, under the conditions stated above of number of maturities and states, there is a unique equilibrium. However, it is no longer possible that the equilibrium be invariant to the sunspot. As stated above, in order for the equilibrium for both the allocations and prices to be invariant to the sunspot, it would have to be the case that the state-contingent prices did not vary with the sunspot. In that case, the noncontingent rates would also be invariant to the sunspot, implying that the matrix $M$ in (20) would not be invertible. In order for an equilibrium to be unique when policy is a target for the term structure, the equilibrium must depend on the sunspot. But there is a (precise) sense in which that dependence can be made arbitrarily small.

4.4 The pure expectations hypothesis and local determinacy

Absence of an arbitrage opportunity between holding a two period bond to maturity and rolling over one period bonds for two periods implies, as in (16),

$$\frac{R^2(s^t)}{R^1(s^t)} = E_t\left[R^1(s^{t+1})\right] + \frac{\text{cov}_t\left[R^1(s^{t+1}), E_{t+1} \frac{u_C(R^1(s^{t+2}))}{P(s^{t+2})}\right]}{E_t\left[\frac{u_C(R^1(s^{t+2}))}{P(s^{t+2})}\right]}.$$  (22)

If the covariance between $R^1(s^{t+1})$ and $E_{t+1} \left[\frac{u_C(R^1(s^{t+2}))}{P(s^{t+2})}\right]$ was always zero, for any process of the price level $P(s^{t+2})$, then $R^2(s^t) = R^1(s^t) E_t \left[R^1(s^{t+1})\right]$, which is the pure expectations hypothesis of the term structure. In this case, targeting the short rates
would also set the long rates. Clearly short and long rates would not be independent targets. Policy on the long rates could not be used to determine the price level, that would be indeterminate.

But, in general, the covariance between \( R(1)^{s+1} \) and \( E_{t+1} \left[ u_C(R(1)^{s+2}) \right] \) is not zero. As shown above, it is a function of policy. Targeting both short and long rates, pins down the covariance, or the term premium. Those are additional restrictions on the process for the price level, that allow to implement uniquely a desirable process for allocations and prices (again, given an initial price level).

It is important to note, that the loglinearization of the equilibrium conditions, sets the term premia to zero, and therefore the expectations hypothesis is always verified. In this sense, loglinearizing the model it is not irrelevant for policy evaluation.

In an attempt to side step the multiplicity problem, McCallum (1981) proposed an interest rate feedback rule such that there is a locally determinate equilibrium\(^{17}\) together with multiple explosive solutions that are disregarded. Our approach is different from this more common approach to implementation (see Woodford, 2005) for two reasons: First, and foremost, we do not loglinearize, and therefore term premia are not automatically set to zero. But there is also a more substantive difference, that we consider the full set of equilibria, and do not select equilibria based on their dynamic properties.

5 An economy with prices set in advance

In a flexible price economy, when policy is conducted with interest rate targets, prices are not pinned down but allocations are. Instead, under sticky prices, setting the path

\[^{17}\text{This means that the linear system of equations that approximates the equilibrium conditions in the neighborhood of a steady state, has a unique solution in that neighborhood and multiple solutions outside that neighborhood.}\]
for the short term nominal interest rate neither pins down prices nor allocations. In
this section, it is shown that the results derived above extend to an environment with
sticky prices, with the main difference that the target of the term structure is used
here to pin down both prices and allocations.

The environment is modified to include price setting restrictions. There is, now, a
continuum of goods, indexed by \( i \in [0, 1] \). Each good \( i \) is produced by a different firm.
The firms are monopolistic competitive and a fraction of them set prices one period in
advance. The remaining firms use contemporaneous information to set prices.

The households have preferences described by (1), where \( C(s^t) \) is the standard
Dixit-Stiglitz aggregator of the consumption of the individual goods, \( C(s^t) = \left[ \int_0^1 c_i(s^t) \frac{\theta}{\theta-1} \right]^{\frac{\theta}{\theta-1}}, \theta > 1 \). The households’ intertemporal and intratemporal conditions on the aggregates
are, as before, (5), (6) and (7).

The government must finance an exogenous path of government purchases, \( \{G(s^t)\}_{t=0}^\infty \),
which is also a Dixit-Stiglitz aggregator of individual \( g_i(s^t) \), with the same elasticity
of substitution as for private consumption, \( \theta \).

The fraction \( 1 - \alpha \) of firms that set prices in advance choose \( p^s_i(s^t) \) to maximize
profits, \( E_{t-1} [Q(s^{t+1}/s^{t-1}) (p^s_i(s^t) y_i(s^t) - W(s^t) n_i(s^t))] \), subject to the production
function, \( y_i(s^t) \leq A t n_i(s^t) \), and the demand function \( y_i(s^t) = \left( \frac{p^s_i(s^t)}{P(s^t)} \right)^{-\theta} Y(s^t) \), de-
derived from expenditure minimization by households and government, where \( y_i(s^t) =
c_i(s^t) + g_i(s^t) \), \( Y(s^t) = C(s^t) + G(s^t) \), and \( P(s^t) \) is the price of one unit of aggregate
consumption, the aggregate price level.

All the sticky price firms set the same price

\[
p^s(s^t) = p^s_i(s^t) = \frac{\theta}{(\theta - 1)} E_{t-1} \left[ \eta(s^{t+1}) \frac{W(s^t)}{A(s^t)} \right], \tag{23}
\]

with \( \eta(s^{t+1}) = \frac{Q(s^{t+1}/s^{t-1}) P(s^t)^{\theta} Y(s^t)}{E_{t-1} [Q(s^{t+1}/s^{t-1}) P(s^t)^{\theta} Y(s^t)]} \). The firms that set the price contempora-
ously also choose a common price,

\[ p^f (s^t) = p^f_i (s^t) = \frac{\theta}{(\theta - 1)} \frac{W (s^t)}{A (s^t)}. \] (24)

These conditions, (23) and (24), can be rewritten using (5), (6) and (7) as

\[ E_{t-1} \left[ \frac{u_C (s^t)}{R^1 (s^t)} P (s^t)^{\theta - 1} A (s^t) (1 - L (s^t)) - \frac{\theta}{(\theta - 1)} u_L (s^t) (1 - L (s^t)) \frac{P (s^t)^{\theta}}{p^* (s^t)} \right] = 0 \] (25)

and

\[ \frac{u_C (s^t)}{R^1 (s^t)} A (s^t) - \frac{\theta}{(\theta - 1)} u_L (s^t) \frac{P (s^t)}{p^f (s^t)} = 0. \] (26)

The equilibrium conditions for \( \{ C (s^t), L (s^t) \}, \{ Q (s^{t+1}/s^t), R^j (s^t), P (s^t) \} \), and \( \{ p^f (s^t), p^* (s^t) \} \) can then be summarized by conditions (6) and (7), (25) and (26), as well as the conditions for the price level in each date and state that can be written as

\[ P (s^t) = \left[ \alpha (p^f (s^t))^{1-\theta} + (1 - \alpha) (p^* (s^t))^{1-\theta} \right]^{\frac{1}{1-\theta}}, \] (27)

and the resource constraints,

\[ [C (s^t) + G (s^t)] \left[ \alpha \left( \frac{p^f (s^t)}{P (s^t)} \right)^{-\theta} + (1 - \alpha) \left( \frac{p^* (s^t)}{P (s^t)} \right)^{-\theta} \right] = A (s^t) N (s^t). \] (28)

If all prices were flexible, \( \alpha = 1 \) and \( p^f (s^t) = P (s^t) \), the intratemporal condition (26) and the resource constraint (10) would determine the allocation, \( C (s^t) \) and \( L (s^t) \), as a function of the short rate \( R^1 (s^t) \), for each date and state. Not so under sticky prices. This is clearer in the other extreme case in which all firms set prices in advance,
\( \alpha = 0 \) and \( p^*(s^t) = P(s^t) \). The intratemporal condition (25) in that case becomes

\[
E_{t-1} \left[ \frac{u_C(s^t)}{R^1(s^t)} A(s^t) \left( 1 - L(s^t) \right) - \frac{\theta}{(\theta - 1)} u_L(s^t) \left( 1 - L(s^t) \right) \right] = 0. \tag{29}
\]

Instead of one equation per state, as in (11), there is now only one equation restricting a conditional average of the marginal rate of substitution and transformation.

### 5.1 Targeting the term structure

As under flexible prices, in this environment, it is possible to use either a target for the state-contingent interest rates or, alternatively, a target for the term structure to implement a unique equilibrium. The no arbitrage conditions between state-contingent bonds and noncontingent bonds of different maturities do not depend on the way prices are set by firms. Those conditions were derived from the first order conditions of the households, regardless of the way prices are set. Since households take prices as given, the no arbitrage conditions are still given by expression (19) and (20).

It follows that proposition 2 holds also in this setting, so that the targeting of the term structure is equivalent to the targeting of the state-contingent returns. Below, in proposition 5, we show that the target of the state-contingent returns implements a unique equilibrium. It follows that there is also a unique equilibrium when policy is a target for the term structure.

**Proposition 5** In an environment with firms that set prices in advance, if the prices of the one-period-ahead, state-contingent nominal assets are set exogenously for every date and state, there is a unique equilibrium for the allocations and prices, given an initial aggregate consumption level.

Proof: Let \( \{Q(s^{t+1}/s^t), t \geq 0\} \) be set exogenously. Then \( \{R^1(s^t), t \geq 0\} \) are determined uniquely by (14). At any \( t \geq 1 \), given \( P(s^{t-1}), C(s^{t-1}) \) and \( L(s^{t-1}) \) there are
Φt intertemporal conditions (7), Φt resource constraints (28), Φt price level conditions (27), Φt−1 price setting conditions (25) and Φt price setting conditions (26). These determine the same number of variables: Φt consumption levels C(s′), Φt levels of leisure L(s′), Φt price levels P(s′), Φt flexible prices p′(s′), and Φt−1 sticky prices p∗(s′).

For t = 0, there is one price level condition (27), one resource constraint (28), one price setting condition (26). The variables are C(s0), L(s0), P(s0) and p′(s0). The sticky prices in period zero, p∗(s0), are exogenous. Given the initial aggregate consumption level, C(s0), the other three variables are determined uniquely. 

Under flexible prices, the target of the state-contingent interest rates, Q(s′/s′−1), determines the nominal interest rate, R1(s′), and, given the nominal interest rate, the allocations, C(s′) and L(s′), are uniquely determined. The distribution of prices across states, P(s′), is determined from (13) for t ≥ 1. Instead, under sticky prices there is no recursiveness in the determination of the equilibrium. Both prices and allocations are determined simultaneously, as described in the proof above. \(^{18}\)

Proposition 5 and proposition 2 imply the following corollary:

**Corollary 6** Suppose a share of firms set prices in advance. Let S_t = \{s_1, s_2, ..., s_n\} and suppose there are nominal noncontingent assets of maturity j = 1, ..., m. Let m ≥ n. If the returns on n of these assets are set exogenously, then, in general, there is a unique equilibrium for the allocations and prices, given an initial level of consumption C(s0).

### 6 A simple example

The economy is now simplified and an example is given. The economy is cashless as discussed in section 4.2. For simplicity, the variables that are a function of the history

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\(^{18}\)When all prices are set in advance, the target of Q(s′/s′−1) across states determines, together with the resource constraint (10), the distribution of allocations across states.
\( s^t \) are indexed by \( t \). The period utility function is \( u(C_t, L_t) = \ln C_t + \phi L_t \) with \( \phi > 1 \).

Government consumption is assumed to be zero, \( G_t = 0 \), and the technology parameter is \( A_t = A_{t-1} \gamma_t \), where \( \gamma_t \) is an independently identically distributed (i.i.d.) random variable with mean \( \gamma \).

**The example with flexible prices** In the case of flexible prices, the intratemporal conditions\(^\text{19} \) are \( C_t = \frac{A_t}{\delta} \), which, together with the resource constraints, \( C_t = A_t (1 - L_t) \), determine consumption and leisure in each date and state. Consumption and leisure do not depend on the nominal interest rate, and, therefore, neither does the (state-contingent) gross real interest rate \( r_{t,t+1} \equiv \frac{u_C(t)}{\beta u_C(t+1)} = \frac{\gamma_{t+1}}{\beta} \).

If \( \gamma_t \) was deterministic and if policy was a target for the short term nominal interest rate, then the equilibrium paths for the price level would be characterized by \( P_{t+1} = \beta R_{t+1} \), \( t \geq 0 \). Given the initial price level, \( P_0 \), the price level in the subsequent periods would be determined recursively by this condition.

But, suppose, instead, that \( \gamma_t \) is stochastic and that there is no other source of uncertainty. And, let policy be a target for the short rate given by \( R_t = \frac{1}{\beta} \pi_t \), with \( \frac{1}{\beta} = E \left[ \frac{1}{\gamma_{t,t+1}} \right] \) and \( \pi_t \geq \frac{\beta}{\Gamma} \). Then, there are multiple equilibrium paths for inflation. In particular, the paths for inflation given by

\[
\frac{P_t}{P_{t+1}} = \frac{1}{\pi_t} + k \left( \gamma_{t+1} - \Gamma \right)
\]

are equilibrium paths for any parameter \( k \) consistent with positive price levels. In one equilibrium, with \( k = 0 \), realized inflation does not vary across states. But for other equilibria, with different \( k \), inflation could vary with the shock, possibly with very high variance.

A target for the state-contingent interest rates guarantees that there is a unique

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\(^{19}\)These conditions are the ones analogous to (11), without the nominal interest rate.
equilibrium (given an initial price level). Suppose policy is a target for $Q_{t,t+1}^{-1}$, with $\frac{1}{R_t} = E_t Q_{t,t+1}$. Now the relevant marginal conditions between any two consecutive states are,

$$\frac{\gamma_{t+1}}{\beta} = Q_{t,t+1}^{-1} \frac{P_t}{P_{t+1}}, \quad t \geq 0. \quad (31)$$

In this case, given $P_0$, the price level in every state is determined uniquely by this condition. One policy could be to have $Q_{t,t+1}^{-1}$ not vary with the shock at $t+1$, satisfying

$$Q_{t,t+1}^{-1} = \frac{\gamma_{t+1}}{\Gamma} R_t^1, \quad \text{for all states at } t+1, \quad (32)$$

in which case the equilibrium would have constant realized inflation across states, $\frac{R_{t+1}}{R_t} = \pi_t$. But, if the state-contingent interest rates were such that $\frac{Q_{t,t+1}^{-1}}{\gamma_{t+1}}$ varied across states, so would realized inflation.

An alternative implementation of a unique equilibrium is with a target for the term structure. Suppose the shock $\gamma_t$, can take only two values $\gamma^l$ and $\gamma^h$, with probabilities $\pi^l$ and $\pi^h$. And suppose the interest rates on the noncontingent bonds were restricted to be time independent, and depended only on the shock. The arbitrage conditions between the noncontingent and contingent interest rates can then be written as

$$\begin{bmatrix} Q_{s,l} \\ Q_{s,h} \end{bmatrix} = M^{-1} \begin{bmatrix} \frac{1}{R^1_t} \\ \frac{1}{R^2_t} \end{bmatrix}, \quad \text{with } M = \begin{bmatrix} \pi^l & \pi^h \\ \frac{\pi^l}{R^1_t} & \frac{\pi^h}{R^1_t} \end{bmatrix},$$

where $R^1_s$ and $R^2_s$ are the one and two period maturity gross interest rates in state $s \in \{l, h\}$, and similarly, $Q_{s,l}^{-1}$ and $Q_{s,h}^{-1}$ are the state-contingent (gross) interest rates from state $s$ to state $l$ and $h$, respectively.

In this economy, a target for the term structure implements a unique equilibrium if the matrix $M$ is invertible. That is the case as long as $R^1_l \neq R^1_h$. This implies that expected inflation must be different in the two states, even if the two values can be arbitrarily close.
The analysis can be redone assuming that the uncertainty is nonfundamental. Instead of the shock to the growth rate $\gamma_t$, the uncertainty now consists in a sunspot $\xi_t$ which is also assumed to be i.i.d. with expected value one and standard deviation $\sigma$. The variable is unrelated to any of the parameters or exogenous variables in the economy, that are assumed to be deterministic. The analysis follows the same steps as before.

Let policy be a target for the short rate. Because, the economy is cashless, consumption and labor are not affected by the sunspot $\xi_t$. They are deterministic, as is the (gross) real interest rate, $r_{t,t+1} = \frac{\gamma}{\beta}$. The equilibria for the price level are described as in the case with the fundamental shock. But, here, the price level can be a function of the sunspot.

As before, the paths for inflation given by

$$\frac{P_t}{P_{t+1}} = \frac{1}{\pi_t} + k (\xi_{t+1} - 1), \quad (33)$$

for any $k$ consistent with positive prices, are possible equilibria for the price level. One equilibrium, with $k = 0$, will not depend on the sunspot.

Again, in this case, a target for the interest rates, $Q_{t,t+1}^{-1}$, on the state-contingent bonds, would implement a unique equilibrium, as is clear from (31) with $\gamma_{t+1} = \gamma$. These state-contingent assets would be paying one unit of money in contingencies that are not fundamental. Possibly, the more interesting policy is to have $Q_{t,t+1}^{-1}$ be the same for all realizations of the sunspot. Condition (32) holds with $\gamma_{t+1} = \gamma$, and the equilibrium would have realized, and expected, inflation be constant across states, $\frac{P_{t+1}}{P_t} = \pi_t$. If, instead, the state-contingent interest rates were to vary with the sunspot, so would inflation.

Also in this case, the targeting of the term structure would be an alternative imple-
mentation. It turns out however that expected inflation would have to be a function of the sunspot, even if that dependence could be made arbitrarily small.

The example with sticky prices  The sticky price environment in section 5 is simplified to include only firms that set prices one period in advance, so that $\alpha = 0$. The elasticity of substitution is assumed to be arbitrarily large, $\theta \to \infty$, so that the average mark up is zero. The economy is cashless, and the period utility function is the same as above, $u(C_t, L_t) = \ln C_t + \phi L_t$. Government consumption is also assumed to be zero.

The intratemporal condition that uses the marginal conditions of the households and the firms is $E_{t-1} [u_C(t) A_t (1 - L_t) - u_L(t) (1 - L_t)] = 0, \ t \geq 1$, which can be written as

$$E_{t-1} \left[ \frac{\phi C_t}{A_t} \right] = 1, \ t \geq 1, \quad (34)$$

restricting the conditional average of consumption for $t \geq 1$. While under flexible prices $\phi C_t = A_t$ for every state, here, $\phi C_t$ is only equal to productivity on average. There are multiple equilibrium processes for consumption sharing the same conditional average.

The intertemporal conditions for the noncontingent short bonds can be written as

$$\frac{1}{R_t} = \frac{P_t}{P_{t+1}} E_t \left[ \frac{\beta C_t}{C_{t+1}} \right], \ t \geq 0. \quad (35)$$

Given a target for the short rate, for each equilibrium process for consumption, there is some inflation rate that satisfies these conditions. Different processes for consumption will correspond to different processes for leisure satisfying the resource constraint, $C_t = A_t (1 - L_t), \ t \geq 0$.

Suppose the only source of uncertainty is productivity, with $\frac{A_{t+1}}{A_t} = \gamma_{t+1}, \ t \geq 0$. As
Before, $\gamma_{t+1}$ is i.i.d. with mean $\gamma$. In this economy, with a target for the short rate, $R_t^1$, there are multiple equilibria for both allocations and inflation. One equilibrium has $\phi C_{t+1} = A_{t+1}, \ t \geq 0$, as under flexible prices. This satisfies (34), and inflation, from (35) is given by $R_t^1 = \frac{\Gamma P_t}{P_{t+1}}, \ t \geq 0$, where $\Gamma^{-1} = E\left[\frac{1}{\gamma_t}\right]$. This is the equilibrium that maximizes welfare. But there are many other equilibria, some with very low welfare.

The paths for consumption given by

$$\frac{\phi C_{t+1}}{A_{t+1}} = 1 + k \left(\gamma_{t+1} - \gamma\right), \ t \geq 0,$$

are also equilibria, for any $k$ consistent with positive consumption and with leisure in the unit interval. These are equilibria with extra variability of the consumption-labor wedge. Inflation, $\frac{P_{t+1}}{P_t}$, is the one that satisfies the intertemporal condition (35), which implies $\frac{1}{R_t^1} = \frac{P_t}{P_{t+1}} E_t \left[\frac{\beta [1+k(\gamma_t - \gamma)]}{\gamma_{t+1} [1+k(\gamma_{t+1} - \gamma)]}\right], \ t \geq 0$.

If instead of targeting the short-term noncontingent rate, policy was a target for the state-contingent interest rates, there would be a unique equilibrium. Suppose policy is a target for $Q_{t,t+1}^{-1}$, that is consistent with the same path for the noncontingent rate, i.e. $\frac{1}{R_t^1} = E_t Q_{t,t+1}$. The intertemporal conditions are now

$$C_{t+1} = E_t Q_{t,t+1} P_t, \ P_{t+1}, \ t \geq 0.$$

Given $C_0$, these conditions together with the intertemporal condition (35) and $\frac{1}{R_t^1} = E_t Q_{t,t+1}$ determine jointly $C_{t+1}$, and (predetermined) inflation $\frac{P_{t+1}}{P_t}$, in every state for $t \geq 0$.

A particular policy, $Q_{t,t+1} = \frac{1}{R_t^1} \frac{P_t}{\gamma_{t+1}}, \ t \geq 0$, implements the first best uniquely, with $\phi C_{t+1} = A_{t+1}, \ t \geq 0$, for $\phi C_0 = A_0$.

That the term structure can be used as an alternative implementation is straightforward from the equivalence of the two alternative targets, the state-contingent rates.
and the term structure. If instead of the productivity shock, or in addition to it, there
was also nonfundamental uncertainty, the analysis in the flexible price case, above,
would go through in this economy with sticky prices.

7 Concluding Remarks

Two main results are obtained in this paper. The first result, of practical interest
for policy, is that a central bank can independently target both short and long-term
nominal interest rates, possibly the whole term structure. This helps explain the
apparent ability of central banks to peg interest rates at different maturities; being
the operations of the Fed and the ECB during the recent financial crisis, of 2008 and
2009, the most striking evidence of it.

The second result is more theoretical. It is shown that setting both short and long-
term nominal interest rates allows to solve the problem of multiplicity of equilibria
associated with uncertainty, that arises when monetary policy is conducted with a
rule for the noncontingent, short-term, nominal interest rate. The result is general in
the context of the model. But, it relies on assumptions that are hard to translate to
features of the actual economy. A necessary condition is that the number of maturities
that are independently targeted equals the number of possible contingencies. But what
are the relevant contingencies in the actual economy?
References


